Magnetically Induced Plasma Rotation

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The extensively studied and controversial question about the origin of the observed fast rotation for magnetically confined plasmas, in particular theta pinches, is briefly reviewed. It is shown that standard MHD equations predict the experimentally observed rotation. The effect is interpreted as a form of the Einstein-de Haas effect. The generally accepted continuum plasma reasoning which denies any appreciable electromagnetic angular momentum transfer to an azimuthally symmetric plasma volume is shown to be too simplified.

I. Introduction

For many years high temperature plasmas have been produced and studied in magnetic compression devices known as theta pinches. From observations of moving irregularities on the plasma cylindrical surface it was early found that the plasma evidently rotated around its axis of symmetry. After losing its rotational symmetry the plasma usually broke into parts, became turbulent and disintegrated [1-6]. The actual rotation is now firmly established by different diagnostic techniques. The direction of rotation is always that of the gyration of a positive ion in the magnetic theta pinch field. The rotation starts simultaneously with the onset or at least in a very early stage of the discharge [7,8] and it leads to such large velocities, up to about 10^7 rev./sec., that even the velocity distribution of emitted thermonuclear neutrons will be shifted [9].

In spite of extensive experimental investigations like the recently published Ref. [10] there is still controversy about the origin of the rotation. An impressive number of mechanisms has been proposed in order to explain how a part of or the whole plasma body can acquire the observed mechanical angular momentum. The theories can all be regarded as consequences of a restriction that the torque causing the rotation must be of mechanical instead of electromagnetic origin if the plasma body is taken to be azimuthally symmetric, incompressible, quasi-neutral, surrounded by vacuum and with no currents flowing in or out of the plasma surface. A purely electromagnetic torque does exist even then but it has been shown to be exceedingly small and totally insufficient to cause the observed rotation [11,12]. The proposed mechanisms therefore rely on the radial [13] or axial [14] division of the plasma into oppositely rotating parts so as to give no net angular momentum, or electrical contact with conducting walls so as to obtain circulating currents [15], or physical plasma contact with the container wall [12], or deviations from azimuthal symmetry [12], or other effects discussed in the comprehensive review paper by Haines [12] published in 1965. Since then Düchs [16] has used numerical methods for taking into account as many of these effects as possible in an extensive study of the effect of applied transverse magnetic fields upon two-dimensional theta pinch dynamics. Benford [10] reports qualitative similarity between his observations and Düchs's numerical results, however, the existence of sufficiently strong transverse fields to cause the observed velocities could not be experimentally proved. The analytical mechanisms listed by Haines [12], including that about the influence of small transverse fields, are discussed but none of them is given support in Benford's paper.

In the following section it will be proved that the two standard MDH equations for charge and mass transport do predict the observed plasma rotation with the correct sign, magnitude and, more significantly, the observed early onset. This theoretically obtained rotation is definitely of electromagnetic origin. Hence, results obtained directly from usual plasma equations seem to be in conflict with that mentioned and apparently very general result [11,12] which states, roughly, that practically no mechanical angular momentum can be imparted to a plasma from the electromagnetic field. Theoretically, the contradiction concerns the symmetry properties of the total matter-plus-field plasma stress tensor, and the main purpose of this paper is to present in Sect. III a solution to it in simple physical terms.

II. Mass Rotation from the MHD Equations

The usual MHD-equations for mass and charge transport are taken to be valid. With standard notations, i.e. e is the ion charge and $\mathbf{V}$ is mass velocity taken to be equal to the average ionic velocity, they are

$$\varrho \frac{d\mathbf{V}}{dt} = \varrho_e \mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla \cdot \mathbf{P}, \tag{1}$$

$$\frac{\mathbf{j}}{\sigma} = \mathbf{E} + \mathbf{V} \times \mathbf{B} + \frac{1}{e\,n_e} \nabla \cdot \mathbf{P}_e - \frac{1}{e\,n_e} \mathbf{j} \times \mathbf{B}. \tag{2}$$

Neglecting displacement currents, the current density $\mathbf{j}$ is related to the velocities $\mathbf{V}$ and $\mathbf{V}_e$ as

$$\mathbf{j} = e\,n_e(\mathbf{V} - \mathbf{V}_e), \tag{3}$$

so that Eq. (2) simply means a balance of the forces acting on the electron gas

$$0 = n_e\,m_e(d\mathbf{V}_e/dt) = -e\,n_e(\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) \\ - \nabla \cdot \mathbf{P}_e + e\,n_e(\mathbf{j}/\sigma) \tag{4}$$

where the last term represents the frictional force between electrons and heavy species.

Consider an arbitrary closed loop, line element $d\mathbf{s}$, attached to the moving plasma mass frame. The loop defines the boundary of a surface, element $d\mathbf{S}$. We apply the identity

$$\frac{d}{dt}\int \mathbf{a} \cdot d\mathbf{S} = \int \frac{\partial \mathbf{a}}{\partial t} d\mathbf{S} \\ + \int \text{div}\,\mathbf{a}\,\mathbf{V} \cdot d\mathbf{S} - \oint \mathbf{V} \times \mathbf{a} \cdot d\mathbf{s} \tag{5}$$

to the function

$$\mathbf{a} = \mathbf{B} + \frac{\varrho}{e\,n_e} \text{curl}\,\mathbf{V} \tag{6}$$

and assume complete or constant fractional ionization so that the divergence integrand in Eq. (5) vanishes, partly also because of the Maxwell equation

$$\text{div}\,\mathbf{B} = 0. \tag{7}$$

The surface integral of the explicit time derivative in Eq. (5) is then transformed into a line integral by means of Stoke's integral theorem and the Maxwell equation

$$\text{curl}\,\mathbf{E} = -\partial \mathbf{B}/\partial t. \tag{8}$$

The next steps are to expand the convective derivative in Eq. (1) as

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + \tfrac{1}{2}\,\text{grad}\,V^2 - \mathbf{V} \times \text{curl}\,\mathbf{V}, \tag{9}$$

eliminate the $\mathbf{j} \times \mathbf{B}$-terms between Eqs. (1) and (2) and then integrate the resulting equation along the loop. A comparison between this expression and the combined Eqs. (5) and (6) directly yields

$$\frac{d}{dt} \oint \frac{\varrho}{e\,n_e} \mathbf{V} \cdot d\mathbf{s} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{S} \tag{10}$$
$$+ \oint \left[\frac{\varrho_e}{e\,n_e}\mathbf{E} - \frac{1}{e\,n_e}\nabla \cdot (\mathbf{P} - \mathbf{P}_e) - \frac{\mathbf{j}}{\sigma} \right] \cdot d\mathbf{s}.$$

The first integrand in the right hand side line integral simply means that a rotational electric field acting on plasma excess charge will cause rotation. Because of the small value of ϱ_e attainable in a plasma this effect is negligible except for extremely thin plasmas [12]. The viscous off-diagonal terms of the pressure tensors may generally not be unimportant but their effects are, like the electron-ion friction term $\mathbf{j}/\sigma$, so as to inhibit, not drive, mass rotation.

A small but in principle possible contribution of angular momentum from the diagonal scalar pressure parts can be found by rewriting, with the aid of the identities (5) and (9) and the Maxwell Eqs. (7) and (8), an almost trivial equation for ionic motion

$$m_i\,n_i \frac{d\mathbf{V}_i}{dt} = e\,n_i(\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) - \text{grad}\,p_i \tag{11}$$

into integral form

$$\frac{d}{dt} \oint \frac{m_i}{e}\mathbf{V}_i \cdot d\mathbf{s} \\ = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{S} - \oint \frac{1}{e\,n_i}\text{grad}\,p_i \cdot d\mathbf{s}. \tag{12}$$

For the usual scalar pressure $p_i = n_i\,k_B\,T_i$ it is easily found that the last integral can attain a non-vanishing value but only for the rather exceptional plasma properties expressed by

$$\text{grad}\,\ln n_i \times \text{grad}\,T_i \neq 0.$$

There are indications [17] that such a direction difference for density and temperature gradients exists for the highly non-adiabatic electron gas in spontaneously magnetic field generating laser plasmas.

Equation (11) and its integral form (12) describe an ion Vlasovfluid plasma [18]. A comparison between Eqs. (10) and (12) shows that the electrons in such a plasma are thought of as a charge-neutralizing and massless fluid. In the very limit of zero in Larmor gyro radius, i.e. for $m_i \rightarrow 0$, Eq. (12) reduces to the wellknown theorem of magnetic

flux conservation in the moving mass frame

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \approx 0 \,. \tag{13}$$

A less extreme form of Eq. (10) or (12) is obtained by retaining a finite plasma mass so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint \frac{\varrho}{e\,n_\mathrm{e}} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s} = - \frac{\mathrm{d}}{\mathrm{d}t} \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \,. \tag{14}$$

This equation expresses a form of the Einstein-de Haas effect, i. e. mass rotation caused by a magnetic flux variation or the reversed case, i. e. magnetic field generation by mass rotation. In the latter case the effect can be referred to as the Barnett effect. By introducing the magnetic vector potential **A**, Eq. (14) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint \left(\frac{\varrho}{e\,n_\mathrm{e}} \boldsymbol{V} + \boldsymbol{A} \right) \cdot \mathrm{d}\boldsymbol{s} = 0 \tag{15}$$

and it means conservation of canonical angular momentum for the combined heavy species, ions and neutrals, of the plasma. It also suggests the rotation of theta pinch plasmas as the effect of the inductive electric field in the moving mass frame acting on the positive ions. The other aspect, the Barnett effect has been shown [19] to explain a remarkably strong magnetic field generating mechanism operating in the plasma ejected from a coaxial plasma gun.

The very high rotation velocities predicted by Eq. (14) should be noted as well as the simultaneous start of rotation with field penetration. A highly ionized plasma will rotate with an angular velocity of up to the order of the ion Larmor gyration frequency when subject to a fast-rising and permeating external field. For light gases and ambient fields of the order 1 Wb/m² the rotation will be of the order $10^6 - 10^7$ revs/sec in agreement with observations on rotating theta-pinch plasmas [1-6].

The equation of electron motion, (4), can with the aid of Eqs. (5), (7), and (8) be rewritten in integral form as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint \left(\frac{1}{e\,n_\mathrm{e}} \nabla \cdot \boldsymbol{\mathrm{P}}_\mathrm{e} - \frac{\boldsymbol{j}}{\sigma} \right) \cdot \mathrm{d}\boldsymbol{s} \tag{16}$$

where the loop now follows the electron motion. When the right hand side is negligible it expresses the Lighthill theorem [20] of flux conservation in the electron gas frame, not mass frame, and it also predicts the existence of extremely strong and thin

flux-preserving electronic surface currents. Their absence in practically all fusion plasma experiments is normally attributed to instabilities and/or anomalous resistance [21]. However, a diffuse and quickly decaying "skin" current is observed during the early implosion stage of very fast theta-pinches. One may then expect from Eq. (14) strong deviations from the often assumed rigid body rotation [22].

III. Electromagnetic Torque on a Plasma

The Einstein-de Haas effect. Eq. (14) or (15), means that an electromagnetic torque leads to mechanical angular momentum. The belief that such a mechanism of any significant magnitude must be impossible originates from calculations [11, 12] based on a simplified continuum description of the plasma. They concern an idealized single-fluid theta-pinch plasma volume which is azimuthally symmetric, globally charge-neutral and which has no in- or outflows of currents through the bounding surface. The total electromagnetic torque can be expressed in the Maxwell stress tensor $\boldsymbol{\mathrm{T}}_\mathrm{e} = (\varepsilon_0 E^2/2 + B^2/2\,\mu_0)\,\boldsymbol{1} - \varepsilon_0 \boldsymbol{E}\boldsymbol{E} - \boldsymbol{B}\boldsymbol{B}/\mu_0$ and the electromagnetic momentum $\boldsymbol{G}_\mathrm{e} = \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}$ as

$$\int \boldsymbol{r} \times (\varrho_\mathrm{e} \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}) \,\mathrm{d}v = - \int \boldsymbol{r} \times \left(\nabla \cdot \boldsymbol{\mathrm{T}}_\mathrm{e} + \frac{\partial \boldsymbol{G}_\mathrm{e}}{\partial t} \right) \mathrm{d}v \,. \tag{17}$$

As it can be shown that the RHS tensor divergence terms vanish upon integration the conclusion is that only the insignificant electromagnetic momentum can drive rotation.

First we critically note that a plasma is a particle system, not a continuum, and the absence of a net torque on a particle system does not imply conservation of angular momentum. (Conservation would require, in addition, exclusively central force particle-particle interactions.) Second, if one sticks to the concept of a plasma as a system of particles with widely varying masses, the bona fide continuum variable [23] is the particle density n so that the proper momentum variable should be the total momentum per particle $\boldsymbol{G}/n$ where $\boldsymbol{G} = \boldsymbol{G}_\mathrm{k} + \boldsymbol{G}_\mathrm{e} = \varrho \boldsymbol{V} + \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}$ and $n = n_\mathrm{e} + n_\mathrm{i}$. Following the Penfield-Haus [23] discussion of stress tensor symmetry, the torque acting on a unit volume of plasma is expressed as

$$n \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\nu} + \boldsymbol{r} \times \boldsymbol{G}/n) = \nabla \cdot (\boldsymbol{\mathrm{T}}^0 \times \boldsymbol{r}) \tag{18}$$

where $\mathbf{T}^0 = \mathbf{T} - \mathbf{V}\,\mathbf{G}$ is the total stress tensor in the moving fluid frame. The laboratory frame stress tensor $\mathbf{T}$ is the sum of $\mathbf{T}_e$, the pressure tensor $\mathbf{P}$ and the kinetic tensor $\mathbf{T}_k = \varrho\,\mathbf{V}\,\mathbf{V}$ where $\mathbf{V}$ is the mass velocity. $\boldsymbol{\nu}$ is the intrinsic angular momentum of the plasma particles. It is negligible as well as the electromagnetic part $\mathbf{G}_e$ of the total momentum flow $\mathbf{G}$ so that $\mathbf{G} = n_e\,m_e\,\mathbf{V}_e + n_i\,m_i\,\mathbf{V}_i$. The RHS of Eq. (18) includes everything which serves to accelerate the plasma into rotation. To see how these fields and forces distribute for a plasma the LSH is split up into one electronic and one ionic part

$$n_e \frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{r} \times m_e\,\mathbf{V}_e) + n_i\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{r} \times m_i\,\mathbf{V}_i)$$
$$= \nabla \cdot (\mathbf{T}^0 \times \mathbf{r}) \,. \quad (19)$$

Observe that the substantive derivative operating on $\mathbf{r}$ equals $\mathbf{V}_e$ in the first term and $\mathbf{V}_i$ in the second so that

$$\mathbf{r} \times n_e\,m_e\,\frac{\mathrm{d}\mathbf{V}_e}{\mathrm{d}t} + \mathbf{r} \times n_i\,m_i\,\frac{\mathrm{d}\mathbf{V}_i}{\mathrm{d}t} = \nabla \cdot (\mathbf{T}^0 \times \mathbf{r}) \,. \quad (20)$$

If one accepts the standard assumption of negligible electron inertia, used e. g. in Eq. (2), it follows that the torque, regardless of origin, only acts on the heavy ions. The agreement with the ion Vlasov fluid model and the Einstein-de Haas effect Eq. (14) is obvious.

Instead of using the correct but somewhat crude condition $m_e = 0$ to obtain this result one may alternatively apply microscopic reasoning and use the proportionality between angular and magnetic moments of charged particles to show that the first and electronic term in Eq. (19) is always small for freely gyrating electrons, i. e. when the Hall parameter exceeds unity, in contrast to the ionic second term. Thus one may consider from the microscopic point of view the discussed plasma field-rotation phenomena, proved to be inherent in the fluid Eqs. (1) and (2), simply as finite ion Larmor radius effects.

It must be noted that the velocity in Eq. (14) and the conclusions derived from this equation refer to the actual macroscopic plasma mass velocity and not any ionic guiding-center velocity which may be quite different. The difficulties associated with analyses based on drift motion equations for guiding-centers can be illustrated by the often overlooked fact that even the vacuum electric field inside the very simplest cylindrical single-turn theta-pinch coil is rather complicated with no azimuthal symmetry and no closed electric field lines [24]. Actually, the rise time of the applied field in most theta-pinch experiments is so short that the microscopic ion motion will be highly non-adiabatic thus invalidating any guiding-center theory.

IV. Summary and Conclusions

The essence of presented derivations is a decoupling of the plasma motion into one electronic and another for the heavy species. This partition requires the $\boldsymbol{j}/\sigma$ friction term in the generalized Ohm's law to be small compared to the moving mass frame electric field which in turn implies strong electronic Hall effect. This condition is normally satisfied by plasmas in intermediate or strong magnetic fields.

It is concluded that magnetically induced plasma rotation can explain an observed rotation of theta pinch plasmas. The effect is electromagnetic, contained in the usual MHD equations, derivable from more general stress tensor reasoning, and it can crudely be described as a finite ion Larmor radius effect which leads to a betatron accelerating mechanism acting selectively on the plasma ions. This agrees with the strong and preferential ion heating, $T_i \gg T_e$, which is the experimentally observed and a most attractive feature of fast theta-pinches. (There is some controversy also about the origin of this temperature inequality, the chief competing theories being impurity bremsstrahlung radiation losses and electronic heat conduction to the ends [9]. It might also be noted that some kind of momentum transfer equipartition between oppositely directed azimuthal electron and ion mass flows would upon thermalization rather lead to $T_i \ll T_e \approx T_i\,m_i/m_e$).

We believe that magnetically induced rotation is a very general feature for magnetized plasmas. We have previously suggested that it accounts for a velocity flow profile distortion at the inlet of MHD power generators [25], regular oscillations in the exhaust flow of certain MPD accelerators [26] and rotation of laser-created plasmas expanding in strong applied magnetic fields [27].

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